Dynamic discrete choice models for transportation

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Abstract

Discrete choice models have received widespread acceptance in transport research over the past three decades, being used in travel demand modeling and behavioral analysis; however, their applications have been mainly developed in a static context. There have been several dynamic models in transportation; but these formulations are not based on dynamic optimization principles and do not allow for changes in external factors. With the continuous and rapid changes in modern societies (i.e. introduction of advanced technologies, aggressive marketing strategies and innovative policies) it is more and more recognized by researchers in various disciplines from economics to social science that choice situations take place in a dynamic environment and that strong interdependencies exist among decisions made at different points in time. Dynamic discrete choice models (DDCMs) describe the behavior of a forward-looking economic agent who chooses between several alternatives repeatedly over time. DDCMs are usually specified as an optimal stopping problem, where agents decide when to make a change in ownership of durable goods or in their behavior. In this paper we present the application of the dynamic formulation to short-medium term vehicle holding decisions.

1. Introduction

Discrete choice models based on Random Utility Maximization (RUM) theory have been of interest to researchers for many years in a variety of disciplines. These methodologies are used to analyze and predict individual choice behavior. Classical formulations assume that utilities are linear, additive and include both individual characteristics and alternative attributes. The multinomial logit (MNL) (Ben-Akiva and Lerman, 1985) model has been the most widely used structure for modeling discrete choices in travel behavior analysis. The nested logit (NL) model (Daly, 1982) relaxes in part MNL model assumptions; it is derived from McFadden's (1978) generalized extreme value (GEV) model. Other relaxations of the MNL model, designed to consider similarity between pairs of alternatives, have been derived from McFadden's GEV model as well. These include the ordered generalized extreme value (OGEV) model (Small, 1987), the paired combinatorial logit (PCL) model (Chu, 1981, 1989) and the cross-nested logit (CNL) model (Wen and Koppelman, 2001; Abbe et al., 2007; Papola, 2000). Non-closed form discrete choice models as Probit (Daganzo et al., 1977) and Mixed logit (McFadden and Train, 2000) have been adopted by researchers to deal with heterogeneity over consumer preferences, correlation across alternatives and state dependency. All these models have been mainly developed in a static context. However, the static framework is limited by the assumption that consumers are not affected by past and future states when choosing their preferred alternative in the present. The gap between discrete choice model and dynamics in individual behavior has spurred various developments that are mainly intended to enrich the basic theory by including in the formulation the changes occurring in the system to be modeled.

A significant portion of the literature focusing on the extension of discrete choice models into a dynamic frame can be found in economics and related fields. In dynamic discrete choice structural models, agents are forward looking and maximize expected inter-temporal payoffs; the consumers is aware of the rapidly evolving nature of product attributes within a given period of time and different products are supposed to be available on the market. Changing prices and improving technologies have been the most visible phenomena in a large number of important new durable goods markets. Although sometimes the future effects are not fully known, or depend on factors that have not yet transpired, the person knows that in the future, he will maximize utility among the alternatives that will available at that time. This knowledge enables consumers to choose the alternative in the current period that maximizes his expected utility over the current and future periods (Train, 2003). As a result, a consumer can either decide to buy the product or to postpone the purchase at each time period.

This dynamic choice behavior has been treated in a series of different research studies and the modeling procedures were applied to various areas. Wolpin (1984) modeled women's fertility and later job search, while Pakes (1986) proposed an application to patent options. John Rust (1987), who first fully formalized the optimal stopping problem and estimated the optimal stopping time to replace a used bus engine, is considered a pioneer in dynamic modeling. In his dynamic version of McFadden's logit model, a single agent was considered, random components were assumed to be additively separable, conditionally independent and extreme value distributed. Berry, Levinsohn and Pakes (1995) – BLP had shown the importance of incorporating consumer heterogeneity for obtaining realistic predictions of

elasticities and welfare but their models were static and did not account for the inter-temporal incentives of market participants. In 2000, Melnikov expanded the engine replacement model and released the BLP limitations to model the decision of whether to buy a printer machine or to postpone the purchase based on the expected evolution of the product quality and price. Lőrincz (2005) added a persistence effect to the optimal stopping model which completed the standard optimal stopping problem. This persistence means that customers who already had a product may choose to upgrade it. For this application, the model not only included the likely future quality of the product, but also the industry evolution. These dynamic economic models are generally applied to evaluate price and elasticities, intertemporal substitution and the welfare gains from industry innovations.

Carranza (forthcoming) examined the market for digital cameras and proposed a logit utility model with one time purchase; the model incorporated fully heterogeneous consumers and extended standard estimation techniques to account for the dynamics in consumers' characteristics. Gowrisankaran and Rysman (2009) also analyzed the importance of dynamics when modeling consumer's preferences over digital camcorder industry products using a panel data set on prices, sales and characteristics. This model was based on an explicit dynamics of consumer behavior and allowed for unobserved product characteristics, repeated purchases, endogenous prices and multiple differentiated products.

In the transportation field, dynamic models are widely used for dynamic network equilibrium (Lam *et al.*, 2006). For transportation demand analysis, a number of dynamic models have been proposed and calibrated but they are not based on dynamic optimization. Landau *et al.* (1981) defined and tested empirically a framework for trip-generation models sensitive to temporal constraints; household decides whether or not to perform a trip for a specific purpose during the day, and in which period is taken. Hirsh *et al*. (1986) estimated a parametric model of dynamic decision-making process for weekly shopping activity behavior. The individual is assumed to proceed from period to period and the observed weekly activity pattern is the outcome of successive decisions. Action plans are then modified on the basis of actual behavior and of the additional information acquired in previous periods. Liu and Mahmassani (1998) calibrated a day-to-day dynamic model of commuters' joint departure time and route switching decisions that takes into account commuters' learning from experience. The analysis provides insight into day-to-day effects of real-time traffic information on user decisions.

More recently, Train (2003) gives the concept of dynamic decision making and describes a two/more-periods model in his book *Qualitative Choice Analysis*. Ben-Akiva and Abou-Zeid (2007) proposed a dynamic framework to model the evolution of latent variables and observed choices over time. Their approach involved the integration of discrete choice with Hidden Markov chains which contained behavioral dynamics such as individuals' plans, well-being states and actions. Shortly after, the methodology of Hidden Markov chains was used again to model dynamic driving behavior (2007). Choudhury (2007) has studied the effects of unobserved plans for four traffic scenarios: freeway lane changing, freeway merging, urban intersection lane choice and urban arterial lane. These dynamic applications of discrete choice model in transportation focused on the evolution of individuals' previous plans and actions but did not consider the changes in external conditions. Therefore, in transportation the development of dynamic discrete choice models has not been as comprehensive as in economics or marketing.

The present paper provides a review of dynamic theory and its application in economics, with a special focus on the combination of behavioral dynamics and discrete choice. Successively, possible applications in transportation are discussed and extensions of existing frameworks proposed. Finally, we present our conclusions and the avenues for future research opportunities in transportation.

2 Dynamic discrete choice structure and Markov Decision Process

2.1 Definitions and basic concepts for dynamic discrete choice models

Dynamic discrete choice models describe the behavior of a forward-looking agent who is supposed to choose among some available alternatives repeatedly over time and to maximize expected inter-temporal payoffs. The parameters in the dynamic function describe agents' preferences and beliefs about technological and institutional constraints, and the whole utility function contains both the static parameters and the transition probabilities. The ultimate objective is to estimate the structural parameters in preferences and state transition probabilities. The factor by which next period's utility are discounted (discount factor β) may be a constant between zero and one or might be estimated together with preferences and transition probabilities.

The application of dynamic discrete choice models in economics are intended for the consumer i to decide whether to buy a product or not at time t, that is the consumer chooses one of J_t products in period t or chooses to postpone buying. From these J_t choices, the consumer chooses the alternative which maximizes the sum of the expected discounted value of utilities at time $t+1$ conditional on the information at time t. Generally, product j is characterized by observed static characteristics x_j, dynamic characteristic y_{jt} (such as price) and unobserved characteristics ξ_j (e.g. technology innovation) Consumer preferences over x_j and y_{jt} are defined respectively by coefficients α_i^x and α_i^y which need to be estimated with ξ_j. It is assumed that x_j and ξ_j stay constant over infinite life of the product. In each period, the consumer obtains a utility from the product that has just been purchased or from the product that was already owned. The utility function of discrete choice from product j purchased at time t can be generalized as

$$u_{ijt} = \alpha_i^x x_j + \alpha_i^y y_{jt} + \xi_j + \varepsilon_{ijt} \tag{2.1}$$

ε_{ijt} is an individual-specific random term depending on the individual i, the product j and the time period t. It is usually assumed that ε_{ijt} is distributed type I extreme value, independent across consumers, products and time.

The consumer i will decide to buy a product at time period t when the maximum utility is greater than a specific utility which will depend on the expected evolution of products' quality and prices in the future. Let $v_{it} = \max_j \{u_{ijt}\}$ denotes the maximum utility consumer i can get from any product purchased at time t. The reservation utility is the value of not purchasing anything at current time period

t and postponing until the next period $t+1$ when the individual evaluates the problem again. The reservation utility could be written as:

$$V(\Omega_{it}) = \beta E\left[\max\{v_{it+1}, V(\Omega_{it+1})\}\middle|\Omega_{it}\right] \tag{2.2}$$

where Ω_{it} is a vector of sufficient statistics for the distribution of v_{it} and its Markov transition probability. The specific settings of $V(\Omega_{it})$ might differ depending on the specific application considered while the estimation methods used are mostly based on Rust's nested fixed point maximum likelihood algorithm. Both specification and estimation will be discussed in the following Sections.

2.2 Theory of dynamics

According to the formulation proposed by John Rust in 1987, any dynamic problem can be formulated as a Markov decision process (MDP) in which two components should be defined at each discrete period t and for each individual: (1) a vector of system state variable s_t and (2) an action or decision variable d_t. The state and action determine current utility $u(s_t, d_t)$ and affect the distribution of the next period's state s_{t+1} via the Markov transition probability $p(s_{t+1}|s_t, d_t)$. In each period t, the individual maximizes the expected utility $V(s) = \max E(\sum_{t=0}^{T} \beta^t u(s_t, d_t)|s_0 = s)$ and decides the optimal decision rule d.[1] In this equation, E denotes expectation with respect to the controlled stochastic process $\{s_t, d_t\}$ and $\beta \in (0,1)$ is the discount factor. At this stage the Bellman's principle of optimality is advocated to obtain the value function. This principle states that an optimal sequence of decisions in a multistage decision process problem has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions. By applying that principle, the value function can be calculated using a recursive procedure:

$$V(s_t) = \max_{d \in D(s_t)} [u(s_t, d_t) + \beta \int V(s_{t+1}) p(ds_{t+1}|s_t, d_t)] \tag{2.3}$$

and the optimal decision rule is obtained from V by finding a value $d(s) \in D(s)$, where $D(s)$ is the set of decisions, that attains the maximum utility in equation (2.3) for each s (Rust, 1994 draft)

$$d_t(s_t) = \arg\max_{d_t \in D(s_t)} [u(s_t, d_t) + \beta \int V_{t+1}(s_{t+1}) p(ds_{t+1}|s_t, d_t)] \tag{2.4}$$

3 Discussion by model type

3.1 Rust optimal stopping problem

3.1.1 Modeling framework

[1] In a probabilistic approach, where all available information is contained in the observation, decision rules are in general based on the conditional probability. A common procedure is to define a cost function that is minimized or maximized by the optimal decision rule.

An early example of dynamic framework for agent decisions is the optimal stopping model proposed by John Rust in 1987 and applied to the problem of bus engine replacement. In this specific case, the optimal stopping rule was defined as "whether or not to replace the current bus engine" in each period. The stochastic dynamic problem formalizes the trade-off between the conflicting objectives of minimizing maintenance costs versus minimizing unexpected engine failures. Rust's framework focuses on two ideas: (1) a "bottom-up" approach for modeling the replacement problem and a (2) "nested fixed point" algorithm for estimating dynamic programming models in the presence of discrete choices.

The bottom-up approach generates replacement investments by aggregating single replacement demand for some specific capital good such as bus engine (Rust aggregates all the models by bus engine type). The demand is the sum of a large number of stochastic processes, each characterized by a decision variable $\{d_t\}$, where $d_t = 1$ if a replacement occurs and 0 otherwise, and by a state variable $\{s_t\}$ which is the mileage cumulated by the bus engine at time *t*. At each time period the agent faces the following discrete decisions: (i) perform normal maintenance on the current bus engine and incur operating cost $c = (s_t, \theta_1)$[2] or (ii) cannibalize the old bus engine for scrap value *P* and install a new bus engine at cost $\overline{P}$ and incur operating cost $c = (0, \theta_1)$. It is also assumed that the mileage travelled each month is exponentially distributed with parameter θ_2. Besides, there are still some variables that can be observed by the agent but not by the econometrician, a solution is to add an error term ε_t to the utility function $u(s_t, d_t, \theta) + \varepsilon_t(d)$ which realizes single period utility value when alternative d is selected and the state variable is x_t, $\theta = \{\theta_1, \theta_2\}$.

Suppose the vector of state variables obeys a Markov process with transition density given by a parameter function $\pi(s_{t+1}, \varepsilon_{t+1} \mid s_t, \varepsilon_t, d_t, \theta)$. The behavioral hypothesis is that agent chooses a decision rule to maximize his expected discounted utility over an infinite horizon where the discount factor $\beta \in [0,1)$. The solution to this optimal stopping problem is given by the recursive Bellman's equation:

$$V_\theta(s_t, \varepsilon_t) = \max_{d_t \in D(x_t)} \left[u(s_t, d_t, \theta) + \varepsilon_t(d_t) + \beta EV_\theta(s_t, \varepsilon_t, d_t) \right] \tag{3.1}$$

where the utility function *u* is given by:

$$u(s_t, d_t, \theta_1) = \begin{cases} -c(s_t, \theta_1) + \varepsilon(0) & if \quad d_t = 0 \\ -\left[\overline{P} - P + c(0, \theta_1)\right] + \varepsilon(1) & if \quad d_t = 1 \end{cases}$$

In function (3.1), $V_\theta(s_t, \varepsilon_t)$ is the maximum expected discounted utility obtained by the agent when the state variable is (s_t, ε_t). The expected function EV_θ is defined by

$$EV_\theta(s_t, \varepsilon_t, d_t) = \int V_\theta(s_{t+1}, \varepsilon_{t+1}) \pi(ds_{t+1}, d\varepsilon_{t+1} \mid s_t, \varepsilon_t, d_t, \theta) \tag{3.2}$$

[2] Costs are in general not directly observable, so they are inferred from observations. In Rust case study a total cost function is estimated with parameter θ_1.

A transition probability defines the stochastic process governing the evolution of the mileage variable $\{s_t\}$:

$$p(s_{t+1} \mid s_t, d_t, \theta_2) = \begin{cases} \theta_2 \exp\{\theta_2(s_{t+1} - s_t)\} & if \quad d_t = 0 \quad s_{t+1} \geq s_t \\ \theta_2 \exp\{\theta_2(s_{t+1})\} & if \quad d_t = 1 \quad s_{t+1} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

With all the functions defined above, we conclude that Section by saying that $\{s_t, d_t\}$ is a realization of a controlled stochastic process whose solution is an optimal decision rule d_t that attains the maximum in Bellman's equation (3.1). The objective is to use the observed data to infer the unknown parameter vector θ.

3.1.2 Estimation

Maximum likelihood is the method used to infer the unknown parameters and to derive the probability density function $L(s_1,...,s_T, d_1,...d_T \mid \theta)$ from the data and to compute the estimate $\hat{\theta}$ which maximizes the likelihood function. Rust (1987) sets a Conditional Independence (CI) Assumption yielding to a simple formula for the likelihood function; the procedure to compute (3.1) is then substantially simplified.

$$(CI) \qquad \pi(s_{t+1}, \varepsilon_{t+1} \mid s_t, \varepsilon_t, d_t, \theta) = p(s_{t+1} \mid s_t, d_t, \theta)\, q(\varepsilon_t \mid s_t, \theta) \tag{3.4}$$

CI limits the pattern of dependence in $\{s_t, \varepsilon_t\}$ in two ways. First, s_{t+1} is a sufficient statistic for ε_{t+1} so that any statistical dependence between ε_t and ε_{t+1} is transmitted entirely through the vector s_{t+1}. Second, the probability density for s_{t+1} depends only on s_t and not on ε_t. If we assume that q yields some specific functional form such as multivariate extreme value distribution, the likelihood function can be written as:

$$L(s_1,...,s_T, d_1,...d_T \mid \theta) = \prod_{t=1}^{T} P(d_t \mid s_t, \theta) P(s_t \mid s_{t-1}, d_{t-1}, \theta) \tag{3.5}$$

Where the conditional choice probability $P(d \mid x, \theta)$, is given by the standard multinomial logit formula:

$$\frac{\exp\{u(s,d,\theta) + \beta EV_\theta(s,d)\}}{\sum_{d' \in D(x)} \exp\{u(s,j,\theta) + \beta EV_\theta(s,j)\}} \tag{3.6}$$

where EV_θ is the fixed point to the contraction mapping $T_\theta(EV_\theta) = EV_\theta$ computed by:

$$EV_\theta(x,d) = T_\theta(EV_\theta)(s,d) \equiv \int \log[\sum_{d' \in D(s')} \exp\{u(s',d',\theta) + \beta EV_\theta(s',d')\}] p(ds',d,\theta) \tag{3.7}$$

T_θ is a contraction mapping and EV_θ is the unique solution to (3.7). To conclude, the nested fixed point optimization finds a θ that maximizes the likelihood function (3.5). Further details about the optimization algorithm can be found in Rust, 1988.

3.2 Melnikov demand model for differentiated durable products

The bus engine replacement problem only describes a single agent choice behavior; this assumption limits the application of dynamic discrete choice models. Another example of dynamic demand framework is the Melnikov's model for computer printers (Melnikov, 2000). The computer hardware market is similar to many other high-technology product markets; the quality rapidly improves over time and product durability impacts the evolution of prices and sales. In Melnikov's model, only one purchase is made, all consumer heterogeneity is captured by a term that is independently distributed across consumers, products and time. A significant difference exists between this approach and Rust model; Melnikov mainly deals with differentiated durable products while Rust models homogenous products only. The framework is divided into three parts: consumer optimal stopping problem, industry evolution and sales dynamics and aggregation.

3.2.1 Consumer optimal stopping problem

The consumer optimal stopping problem gives a general formulation of the choice decision. In each period t, consumer i has two options, $S_{it} = \{0,1\}$. $s_{it} = 0$ means i does not own any product at t; $s_{it} = 1$ otherwise; in the latter case consumers are out of market. In each period t consumers who have no product either choose to buy one of the products j or to postpone the purchase until the optimal time. If the consumer buys a product the terminal payoff, which is the utility gained when the consumer decides to buy, is:

$$u_{ijt} = f(x_j, y_{jt}; \theta_i) + \varepsilon_{ijt} \tag{3.8}$$

where x_j is a vector of static product attributes for product j, y_{jt} is a vector of dynamic characteristics such as price for product j at time t, θ_i is a vector of parameters for homogenous consumer preferences over *x* and *y*, so it can be simplified as θ under the author's assumption; random terms ε_{ijt} are individual-specific random utility components of *J*-dimensional random vector ε which are assumed to be *iid* amongst individuals and periods and follow a generalized extreme value (GEV) distribution. Based on the description above, u_{ijt} are therefore *iid* amongst individuals as well. We can neglect the different individuals in (3.8) because of the assumption of homogeneity and decompose it as $u_{jt} = \delta_{jt} + \varepsilon_{jt}$, where δ_{jt} is the mean utility $E[u_{ijt}]$.

Generally, the consumer makes the decision following two steps: first he chooses j_t^* that maximizes the utility from set J and then he decides whether to buy or to postpone the purchase until the next period. j_t^* is the product which contributes the maximum utility and set J includes all the products j available to the consumer. This optimal stopping problem can be generated as the following formula:

$$D(u_{1t},...,u_{Jt}) = \max_{\tau}\left\{\sum_{k=t}^{\tau-1}\beta^{k-t}c + \beta^{\tau-t}E[\max_{j\in J} u_{j\tau}]\right\} \tag{3.9}$$

where β is a common discount factor; c is the utility payoff and E denotes a conditional expectation. Let $v_t = \max_{j\in J} u_{j\tau}$ and v_t has type I extreme value (Gumbel) distribution according to the described assumption about ε_{ijt}. The distribution of v_t is Gumbel distributed with a scale factor 1 (because of the assumption defined in this paper), so we have

$$F_v(z;r_t) = \exp(-e^{-(z-r_t)}) \tag{3.10}$$

where r_t in formula (3.10) is the mode of the distribution of v_t. The consumer's decision can be finally transformed from (3.9) into:

$$D(v_t, c_t) = \max\left\{v_t, c + \beta\ E[D(v_{t+1})]\right\} \tag{3.11}$$

3.2.2 Industry evolution

Melnikov's model contains a very important factor r_t which characterizes the distribution of the maximum utility; it represents the evolution of the industry and it is formulated as the mode of the Gumbel distribution of v_t. It is also assumed that the evolution of the mean utility can be characterized by a homogenous Markov process with transition density $\Phi(r_{t+1}|r_t,\theta_r)$. Besides, r_t here follows a diffusion process defined by:

$$r_{t+1} = \mu(r_t) + \sigma(r_t)\upsilon_{t+1} \tag{3.12}$$

where υ_t are assumed to be i.i.d. standard normals $N(0,1)$. $\mu(r)$ and $\sigma(r)$ are continuous and almost everywhere differentiable and $\mu(r) \geq r$. The diffusion process can be expressed by means of different formulations. Here r_t has a homoschedastic random walk with drift, $r_{t+1} = r_t + \gamma + \sigma\upsilon$ (where $\gamma \geq 0$). In this case, the Bellman equation (3.11) becomes:

$$D(v_t, r_t) = \max\left\{v_t, c + \beta\ E[D(v_{t+1}(r_{t+1}))|r_t]\right\} \tag{3.13}$$

where v_t has Gumbel distribution with mode r_t. Meanwhile, the stopping set is $\Gamma(r) = \left\{v \middle| v \geq c + \beta E\left[D(.)|r\right]\right\}$ and it is convenient to define $W(r) = c + \beta E\left[D(v_{t+1}(r_{t+1}))|r_t\right]$ as the reservation utility.

3.2.3 Sales dynamics and aggregation

- **Demand structure**

The dynamics of the demand structure is determined by the probability of postponing the purchase, which the author denotes as:

$$\pi_{0t}(r_t) = P\left\{S_{i,t+1} = 0 \middle| S_{it} = 0, r_t\right\} = F_v(W(r_t), r_t) = \exp\left(-\exp\left(-\left(W(r_t) - r_r\right)\right)\right) \tag{3.15}$$

The probability of buying the product is defined as the individual hazard rate of the product adoption, $h\left(r_r\right) = 1 - \pi_{0t}(r_t)$. One important issue in this Section is the calculation of the hazard rate with equation (3.15). By setting $Y(r_t) = W(r_t) - r_t$ and by combining (3.12), (3.13), (3.14), $Y(r_t)$ can be integrated as:

$$Y(r_t) = c + \beta\mu(r) - r + \beta\int_{-\infty}^{\infty} E\left[\max(\varepsilon, Y(r_{t+1}))\right]\phi\left(\frac{z - \mu(r)}{\sigma(r)}\right)dz \tag{3.16}$$

We recall here that equation (3.12) and r_t's random walk with drift, $Y(r_t)$ is obtained from (3.16). Thus, the hazard rate h can be computed from (3.15).

- **Aggregation**

The transition of consumer state can be presented by a Markov matrix $H : \{0,1\} \to \{0,1\}$:

$$H_1(r_t) = \begin{bmatrix} \pi_{0t}(r_t) & h(r_t) \\ 0 & 1 \end{bmatrix} \tag{3.17}$$

Rather than using Rust nested fixed point maximum likelihood algorithm, Melnikov uses an easier three-stage method to estimate the models that includes: (1) identifying static parameters by OLS, (2) using maximum likelihood to get parameters γ and σ from transition density $\Phi(r_{t+1} | r_t, \theta_r)$, and (3) estimating the remaining parameters by fitting predicted sales to the data with the moment condition. This method is based on the assumption that sales of product j can be aggregated, total market size is known and that the consumers are homogeneous.

3.3 Computer server choice model with persistence effect

Lőrincz's paper incorporates a persistence effect into the Melnikov optimal stopping problem (Lőrincz, 2005). If the consumer already has one product, he can upgrade it without getting rid of the old one. Hence, besides deciding about the optimal time to buy a product, the consumer who already has a product can choose between simply using the original product and specifically upgrading its format. Overall, this model is built on three principles: product differentiation, optimal stopping problem and persistence effect. In this example, the model is applied to low-end server computers where formats are represented by operating systems (OSs). Since reliability and security are essential characteristics for servers, upgrades of OSs often need to be carried out. In particular, a dynamic nested logit model is estimated by Lőrincz, where nests are represented by different operating systems.

3.3.1 General dynamic nested logit model

Lőrincz represents the evolution of the state vector by a Markov-transition probability and models the problem by using the Bellman equation (2.3). In this case, the choice set $J(s)$ is partitioned into G +1 mutually exclusive subsets: $J(s) = \bigcup_{g=0}^{G} g(s)$. The subset $g = 0$ means that customers are not buying

any product. The other G subsets correspond to different OSs which are nested. The state is composed of three elements: x, y and ε. x is a set of product specific state variables such as characteristics and price; y is the customer specific state variable observed by econometrician and $y \in \{0,1,...,G\}$. $y=0$ indicates that the customer does not own anything at the beginning of the current period. $y=g$ indicates that the product owned currently belongs to nest g. This latter specification differentiates this approach from the Melnikov's formulation where only states $y=0$ (not owning a product) and break-down probability are considered.

Different utilities need to be specified depending on the conditions y and g:

In case 1, $y=0$ and $g=0$ the customer owns nothing and does not buy, $u_j = c + \varepsilon_0$. The constant c is a payoff.

In case 2, $y=0$ and $g \in \{1,...,G\}$, $j \in g$ the customer owns nothing but buys one from nest g. Payoff is then the sum of a product specific value $u_j = x_j \gamma_g + \varepsilon_g + (1-\sigma_g)\varepsilon_j$ where γ_g is a vector of parameters and $\sigma_g \in (0,1)$ governs correlation in nest g. The terms ε_g and ε_j represent the heterogeneity of nests and products within nests respectively; they are *iid* across nests and periods with extreme value distributions.

In case 3, $y \in \{1,...,G\}$ and $g=0$ the customer does not buy anything when he already owns one product. So he gets a format specific 'continuation value' c_y. $u_j = c_y + \varepsilon_0^u$.

In case 4, $y \in \{1,...,G\}$ and $g \in \{1,...,G\}$, $j \in g = y$ the customer already has a product and decides to upgrade it. So the customer chooses an alternative j from the upgrade nest y of the original product. $u_j = x_j^u \gamma_g^u + \varepsilon_g^u + (1-\sigma_g^u)\varepsilon_j^u$.

Some assumptions are given. ε_0 and ε_0^u are *iid* across all alternatives and periods with extreme value. $\varepsilon_g + (1-\sigma_g)\varepsilon_j$ and ε_g are *iid* across nests and periods with extreme value, that is the same as $\varepsilon_g^u + (1-\sigma_g^u)\varepsilon_j^u$ and ε_g^u.

Then, transition probabilities are specified as following:

$$p(x_{t+1}, y_{t+1}, \varepsilon_{t+1} | x_t, y_t, \varepsilon_t, j) = h(\varepsilon_{t+1} | x_{t+1}, y_{t+1}) f(x_{t+1} | x_t) l(y_{t+1} | y_t, j) \quad (3.18)$$

3.3.2 Estimation of the simplified dynamic nest logit model

The customer is supposed to choose between nests; this assumption reduces the state and choice dimensional space. Since the specific product index j is identified by its nest g, the author replaces j by g in the transition probability function of state y. So in equation (3.18) $l(y_{t+1} | y_t, j)$ can be changed into $l(y_{t+1} | y_t, g)$ while equation (2.1) becomes:

$$V(s_t) = \max_{j \in J(s)} [u_j(s_t) + \beta \int V(s_{t+1}) p(ds_{t+1}, g)].$$

Therefore, it is assumed that the formats of all products belonging to the same nest g are the same and that customer specific persistence effect is carried out through time by the format but not by the product itself. The model is estimated following three steps. First, specify static conditional logit models of within nest choices are estimated; second, the transition probabilities for the models' inclusive values are calculated; then a dynamic logit model of choice between nests including the results from the last two steps is calibrated. More technical details can be found in Lőrincz's paper (2005).

3.4 Dynamic durable goods demand with consumer heterogeneity

In previous examples, consumers are assumed to be homogeneous and randomly *iid*. Under this assumption the parameters of the static problem can be estimated separately from the dynamic one. When extending the original technique to fully heterogeneous consumer problems, the integration of the individual demand function over the distribution of consumers' characteristics is needed. In this context, Carranza (2006, forthcoming) models digital camera demand by using dynamic framework similar to those described in previous Sections but incorporates fully heterogeneous consumers into a reduced form of the participation probability. The author is then able to estimate the joint distribution of consumers' preferences and the parameters associated to the participation function which is based on the observed number of purchases.

3.4.1 A dynamic model of demand

Suppose individual i buys product j, the lifetime utility of this purchase is:

$$u_{ijt} = \xi_{ij} + \alpha_i^x x_j - \alpha_i^p p_{jt} + \varepsilon_{ijt} \tag{3.21}$$

As in the framework presented in Section 2, ξ_{ij} is an unobserved product attribute common to all consumers who purchase product *j* at time *t*; p_{jt} is the price of product *j* at time t and x_j is the vector of observed static characteristics of product j. Preference parameters $(\xi_{ij}, \alpha_i^x, \alpha_i^p)$ vary across consumers. The author lets $\xi_{ij} = \xi_j + \sigma_\xi e_{i\xi}$, $\alpha_i^x = \alpha^x + \sigma_x e_{ix}$ and $\alpha^p{}_i = \alpha^p + \sigma_p e_{ip}$, where e_i is drawn from a know *iid* distribution F_e. So 3.23 can be rewritten as:

$$\begin{aligned} u_{ijt} &= (\xi_j + \alpha^x x_j - \alpha^p p_{jt}) + (\sigma_\xi e_{i\xi} + \sigma_x e_{ix} x_j - \sigma_p e_{ip} p_{jt}) + \varepsilon_{ijt} \\ &= \delta_{jt}(x_j, p_{jt}; \theta_0, \xi_j) + \mu_{ijt}(x_j, p_{jt}; \theta_1, e_i) + \varepsilon_{ijt} \end{aligned} \tag{3.22}$$

In this formula, $\theta_0 = (\alpha^x, \alpha^p)$, $\theta_1 = (\sigma_\xi, \sigma_x, \sigma_p)$ and $e_i = (e_{i\xi}, e_{ix}, e_{ip})$. The utility function has mean δ_{jt} which is common to all consumers, variance μ_{ijt} which captures the variability of tastes across consumers and an idiosyncratic product-consumer random component ε_{ijt}.

The reservation utility of the consumer is the value of not purchasing anything at time t and waiting until the next period to decide. The problem can be formulated as:

$$W(S_{it}) = 0 + \beta E[\max\{v_{it+1}, W(S_{it+1})\} | S_{it}] \tag{3.23}$$

Let $v_{it} = \max_j \{u_{ijt}(.)\}$ be the maximum utility consumer i can get from any product purchased at t and let assume that its distribution $F_{v_{it}}$ is known (recall that in Melnikov case $F_{v_{it}}$ is GEV distributed). The probability that the consumer buys any product at time t can be expressed as a hazard rate (see 3.2.2) and obtained from the known distribution of v_{it}:

$$\Pr(purchase) \equiv h_{it}(S_{it}) = P[v_{it} > W(S_{it})] = 1 - F_{v_{it}}(W(S_{it})) \tag{3.24}$$

The solution of the dynamic optimization problem is a participation function h_{it} that depends on the state vector S_{it}. It is also assumed that ε_{ijt} have an independent extreme value distribution. According to Melnikov's deduction (Melnikov, 2000-see appendix), v_{it} has extreme value distribution with mode r_{it}:

$$r_{it}(.) = \log[\sum_{k \in J_t} \exp(\delta_{kt}(.) + \mu_{ikt}(.))] \tag{3.25}$$

Since v_{it} is assumed to be Markovian, state S_{it} in formula (3.24) and (3.25) can be replaced by r_{it}. Then the specific product purchase probability is:

$$h_{jit}(., e_i) = h_{it}(r_{it}(., e_i)) \frac{\exp(\delta_{jt}(.) + \mu_{ijt}(., e_i))}{\exp(r_{it}(., e_i))}. \tag{3.26}$$

3.4.2 Estimation

To obtain the predicted market share for product j, (3.26) has to be integrated across consumers which can be based on the distribution of F_ε.

$$s_{jt}(\theta_0, \theta_1) = \int [h_t(r_t(\theta_0, \theta_1, e)) \frac{\exp(\delta_{jt}(\theta_0) + \mu_{jt}(\theta_0, \theta_1, e))}{\exp(r_t(\theta_0, \theta_1, e))}] dF_e \tag{3.27}$$

$\theta = (\theta_0, \theta_1)$ can be obtained by equating the predicted and observed demand but the premise is the observed demand for product j at t and market size are known, that is

$$M_t s_{jt}(\theta_0, \theta_1) = Q_{jt} \tag{3.28}$$

where Q_{jt} is the observed demand for product j at t and M_t the market size. The integration of (3.27) can be simplified by using simulation techniques.

3.5 Dynamic durable goods demand with repeat purchases

Carranza's model incorporates consumer heterogeneity into differentiated product demands but does not account for repeat purchases. Gowrisankaran and Rysman (2009) generate a dynamic model of consumer preference for the digital camcorder. It allows for unobserved product characteristics, repeat purchases, endogenous prices and differentiated products.

It is assumed that a consumer who purchases product j at t would receive a net flow utility $u_{ijt} = \delta_{ijt}^f - \alpha_i^p \ln(p_{jt}) + \varepsilon_{ijt}$, where $\delta_{ijt}^f = \alpha_i^x x_{jt} + \xi_{jt}$. δ_{ijt}^f is the gross flow utility from product j

purchased at time t. x_{jt} is observed characteristics and ξ_{jt} is unobserved; p_{jt} is price; ε_{ijt} is an idiosyncratic unobservable parameters. Let Ω_t denote current product attributes and it evolves according to the Markov process $P(\Omega_{t+1}|\Omega_t)$. The author defines a consumer who does not purchase a new product at time t has net flow utility as well: $u_{i0t} = \delta_{i0t}^f + \varepsilon_{i0t}$. Then the value function could be $V(\varepsilon_{i.t}, \delta_{i0t}^f, \Omega_t)$ and the expectation of the value function is $EV_i(\delta_{i0t}^f, \Omega_t) = \int_{e_{i.t}} V(\varepsilon_{i.t}, \delta_{i0t}^f, \Omega_t) dP_\varepsilon$. $\varepsilon_{i.t}$ is *iid* and it satisfies the conditional independence assumption in Rust's 1987 paper. Bellman equation is represented as:

$$V_i(\varepsilon_{i.t}, \delta_{i0t}^f, \Omega_t) = \max\left\{u_{i0t} + \beta E[EV_i(\delta_{i0t}^f, \Omega_{t+1})|\Omega_t], \max_{j=1,\dots,J_t}\left\{u_{ijt} + \beta E[EV_i(\delta_{ijt}^f, \Omega_{t+1})|\Omega_t]\right\}\right\} \quad (3.31)$$

The large dimensionality of Ω_t might induce serious computational problems. Therefore, the author substitute Ω_t with a scalar variable, the logit inclusive value of purchasing in time t:

$$\delta_{it}(\Omega_t) = \ln(\sum_{j=1,\dots,J_t} \exp(\delta_{ijt}(\Omega_t))) \quad (3.32)$$

Besides, there is a main simplifying assumption, the logit inclusive value depends only on the current logit inclusive value that is termed Inclusive Value Sufficiency. This assumption indicates that if two states have the same inclusive value δ_{it} for consumer i at current time t, they have the same distribution of inclusive value for this consumer for the future time. The simplification from this assumption is represented in this formula:

$$EV_i(\delta_{i0t}^f, \delta_{it}, E[\delta_{it+1}, \delta_{it+2,\dots}|\Omega_t]) = EV_i(\delta_{i0t}^f, \delta_{it}) \quad (3.33)$$

To specify the density $P(\delta_{i,t+1}|\delta_{it})$ a simple function is assumed with linear autoregressive specification with drift $\delta_{i,t+1} = \gamma_{1i} + \gamma_{2i}\delta_{it} + u_{it}$,where u_{it} is normally distributed with mean 0 and γ_{1i}, γ_{2i} are parameters.

The estimation algorithm includes three levels of optimization. The inner loop evaluates the predicted market shares as a function of $\overline{\delta}_{jt}^f$ and parameters by solving the consumer dynamic programming problem for the simulated consumers and then integrating across consumer types. The middle loop performs a fixed point equation and iterates until the new and old $\overline{\delta}_{jt}^f$ converge. The outer loop is a search over the parameters. Details can be referred from this paper (Gowrisankaran and Rysman, 2009).

The model allows for consumers' repeat purchases but does not introduce any new parameters over the static model. Other assumptions include: durable goods do not wear out; there is no resale market for them; and there are no households with more than one good at the same type. Therefore, the second purchased good will only have new features that are observed and results to be very different from previous good's type.

3.6 Summary of dynamic demand models in economics

We finally compare the five dynamic models and we present a summary in Table 1; we include authors, dates, data, case description, main formulation and estimation results.

Rust's optimal stopping problem provides the basic model framework and the estimation method for the dynamic discrete choice models developed later in the literature. It is a single agent problem describing the decision of time to make one purchase over a set of products with homogenous attributes (bus engines with different models). The estimation method is the nested fixed point algorithm that computes the maximum likelihood estimates and reduces the computational burden of solving the contraction fixed point. Melnikov's dynamic demand model of computer printers contains the concept that product quality rapidly improves over time and the product durability impacts the evolution of prices and sales. Same as Rust's example, only one purchase is made and all consumer heterogeneity is captured by a term that is independently distributed across consumers, products and time. The difference is that it deals with differentiated durable products rather than homogenous products. The estimation method is a three-stage procedure that replaces the more complicated nested fixed point maximum likelihood algorithm. Then Lőrincz's model extends Melnikov's optimal stopping problem with a persistent effect. The consumer can choose to upgrade the product instead of getting rid of it. Given that different product alternatives and two conditions are considered: without a product (when alternatives are not to buy and to buy a new product) and with the current product (when alternatives are not to upgrade and to upgrade the owned product), thus the decision problem in this case is specified as a dynamic nested logit model. The estimation follows a sequential procedure with three steps. Carranza incorporates fully heterogeneous consumers into a reduced form of the participation probability for a digital camera demand problem. He estimates the joint distribution of consumers' preference and parameters of the participation function which is based on the observed number of purchases. The distribution of preference is defined as a continuous parametric distribution. The complicated integration across consumers in the estimation part needs simulation. Gowrisankaran and Rysman's dynamic model for digital camcorder demand allows for repeated purchases. The estimation algorithm includes three levels of optimization and the estimation could be simplified only with strong assumptions.

All the five models proposed are estimated on retrospective panels of aggregate data. In three cases, data have been recorded on a monthly basis and only in one case each quarter; the total observation period varies from two years to ten years.

To conclude, it is necessary to mention software availability and programming skills needed to implement DDCMs. On that regards, the information are rather limited; it can be said that Rust has developed his own code on GAUSS (Aptech, 2009). The software is available on Rust's website but a password is required. It is reasonable to assume that all the later models have been developed under GAUSS as well; although no precise indications are given in the papers reviewed.

Table 1 Comparison of the five dynamic models

Name	Bus engine replacement	Computer printer demand	Low-end computer server demand with persistence effects	Digital camera demand	Digital camcorder demand
Author (year)	John Rust (1987)	Oleg Melnikov (2000)	Szabolcs Lőrincz (2005)	Juan Esteban Carranza (2006)	Gautam Gowrisankaran & Marc Rysman (2009)
Data	ten years of monthly data	monthly data on computers' sales	quarterly data on quantities, prices	a panel of sales, prices and	monthly level for 378 models and

	on bus mileage and engine replacements for 104 buses	and prices. 462 models from 27 manufacturers. Jan 1998- Jan 1999	and technical characteristics of servers. 1996-2001	characteristics of digital cameras. 1998-2001	11 brands, on average price, other observable characteristics. Mar 2000- May 2006
Characteristics	Single agent, one purchase, homogenous attributes.	Homogeneous consumers with one purchase, differentiated durable products. Potential market size is required.	Homogeneous consumers with one purchase, differentiated servers and upgraded formats.	Fully heterogeneous consumers and differentiated durable products. Potential market size is required.	Repeat purchases, heterogeneous consumers and differentiated products.
Main formula	Describe recursively by Bellman's principle of optimality	Consumers' purchase is an optimal stopping problem including a hazard rate product adoption.	A nested logit model is used to describe the unobserved heterogeneity term.	The identification of the participation function is based on the observation over time.	Bellman's principle of optimality with logit inclusive value of purchasing at a given time.
Estimation method	Nested fixed-point maximum likelihood algorithm.	A nested three-step method that allows for sequential parameters with aggregate data from short time series.	Sequential procedure that estimates transition probabilities and dynamic logit model of choice between nests.	Integrating across consumers to obtain the market demand for each product. Estimate the parameter vector by equating the predicted and observed demand.	Three levels of non-linear optimizations: a search over the parameters; a fixed point optimization; calculation of predicted market shares.

4 Applications in transportation: dynamic model of car ownership

With the rapidly innovating technology nowadays, the study of the acceptance of new vehicle types is very important, for example the uptake of electric, hybrid, bio-fuel cars in the future market. While advanced products proposed by the car industry seem to be promising, they must overcome certain obstacles before they will be competitive in the marketplace. There are three main barriers to their widespread use: cost, infrastructure, and performance. The classical static context in which choices are supposed to happen should be replaced by a dynamic process able to capture interdependencies among decisions made at different points in time. In this Section, we focus on possible applications of dynamic discrete choice models to car ownership, for short and medium-term planning (number of cars and type to own or to purchase) and in particular on the potential of advanced technology (rapidly changing over time) on individual preferences and market evolution**.**

In transportation there is a general agreement about the necessity to account for dynamic effects in car ownership modeling in order to incorporate inter-temporal (uncertainty of financial position in the future) and intra-temporal (acquired taste for a certain lifestyle) dimensions. Nobile et al. (1996) estimated a random effect multinomial probit model for car ownership using panel data drawn from several waves of survey held between 1985 and 1988 in the Netherlands. The model accommodates both intra-temporal and inter-temporal correlation; the intra-temporal correlation is accommodated by allowing a general

form for the error term covariance matrix, while the inter-temporal correlation is captured by unobserved individual specific time invariant attributes also referred to as household heterogeneity. Hanly and Dargay (2000) also used panel data to analyze car ownership for Great Britain. They indicated that the study is based on dynamic discrete choice model and the state variable of current period is influenced by the state in the past. However, the state of each period is only represented by the household car ownership but not by exogenous attributes. Panel data has some limitations, the size and representativeness of the samples decline over time due to attrition, so the data sets are often inferior to the available cross-section data. A pseudo-panel is viewed as an alternative to panel data as it is an artificial panel based on cohort averages of repeated cross-sections. Dargay and Vythoulkas (1999) used the pseudo-panel dataset of 5-year cohorts constructed from repeated cross-section data contained in the UK Family Expenditure Survey. However, one limitation of pseudo-panel data is that averaging over cohorts transforms discrete values of variables into cohort means, therefore individuals' information gets lost.

Dynamic in car ownership has been modeled by means of duration models; this class of models are usually specified to predict when a household will make a vehicle transaction. In 1996, De Jong calibrated a car ownership model system containing four modules: (1) vehicle holding duration, (2) choice of vehicle type, (3) regression equations for annual kilometrage and (4) fuel efficiency. The duration models use continuous time and stochastic models; it is based on a hazard function which gives the probability of exit from the state immediately after a certain time t. The factors influencing the duration decision were attributes of the previous car, social-economic attributes and variables related to the car market. The vehicle type choice model was estimated as a logit. This model accounted for loyalty to certain car brand and engine size and included dummy variables comparing previous and new car characteristics. It should be noted that duration and type choice models were not estimated as a joint model (with correlated error term). The De Jong model was later extended by accounting for car disposal without replacement and constitutes the Dutch Dynamic Vehicle Transaction Model (DVTM, 1993-1995). The most recent dynamic vehicle duration models are in Rashidi, Mohammadian and Koppelman (2009). Using the panel data collected in Seattle and its surrounding areas, which includes 10 waves (from 1989 to 2002), they modeled a system of hazard-based equations in which timing of residential relocation, job relocation and vehicle transaction were the endogenous variables. Timings of the endogenous variables were estimated by a hazard-based duration formulation in a system of simultaneous equations.

Table 2 Comparison of types of dynamic car ownership models

Models	Panel models	Pseudo panel	Dynamic car transaction models (duration model)
Author (year)	Nobile et al. (1996) Hanly and Dargay (2000)	Dargay and Vythoulkas (1999)	De Jong (1996) Rashidi, Mohammadian and Koppelman (2009)
Level of aggregation	Disaggregate	Aggregate	Disaggregate
Long or short run forecasts	Short and long	Short and long	Short & medium

Theory	Can be based on random utility or lifetime utility theory	Weak links with random utility theory	Can be based on random utility
Data requirements	Very heavy	Moderate	Very heavy
Car types	Very limited number of car types possible, but could be combined with a type choice model	Very limited number of car types possible	Very limited number in duration model, but very many in car type choice model
Limitation	The state of each period is only represented by the household car ownership but not by exogenous attributes The size and the representativeness of the samples decline over time due to attrition	Individuals' information gets lost when averaging over cohorts and when transforming discrete values of variables into cohort means	The factors are attributes of the previous car, social-economic attributes and variables related to the car market but without including the market evolution in the future

A dynamic framework seems therefore necessary when modeling consumer demand that explicitly accounts for consumers' expectations of future vehicle quality, evolving markets and consumers' outflow from the car market.[3] We propose here to formalize the timing of consumers' purchases as an optimal stopping problem where the agent must decide on the optimal time of purchase. The model frame might be further enriched by modeling the choice from a set of differentiated vehicles whose quality changes stochastically over time. The possibility to introduce persistence effects, through maintenance and upgrading of the current vehicle, will be also possible. Consumer heterogeneity into differentiated product demands and repeat purchases are also desirable in dynamic vehicles model systems.

When formulating the dynamic car ownership model, we start by defining the states space S. We consider a consumer set $I = \{1,...,M\}$, where each consumer $i \in I$ can be in one of two possible states at each time period t. The two states are: $s_{it} = 0$, consumer i is in the market; $s_{it} = 1$, consumer i is out of the market. "In the market" means that the individual considers buying a car no matter whether he/she currently owns a car. If the individual does not own a car, it is quite possible he/she considers to buy one; if he/she does own a car but with some problematic condition (or plan to sell the previous car), he/she can also consider to replace it or buy an extra car. "Out of the market" means the individual does not consider buying a car at all. In each time period t, consumer i in state $s_{it} = 0$ has two options: to buy one product j and obtain a terminal period payoff u_{ijt} and to postpone and obtain a one-period payoff $c_{it}(x_{it}, q_{it}, \theta_i, \alpha_i)$,which is a function of individual i's attributes and the characteristics of current car owned by i. x_{it} is a vector of attributes for individual i at time t, e.g., sex, education, income, age, etc., and q_{it} is the vector of characteristics of current product owned by this individual. θ_i and α_i are parameters vectors for x_{it} and q_{it} respectively. The payoff u_{ijt} is expressed as a random utility function

[3] Consumer outflow from the car market can be handled in duration models. What these models lack is the theoretical background in dynamic optimization.

$u_{ijt} = U(x_{it}, d_j, y_{jt}, \theta_i, \gamma_i, \lambda_i, \varepsilon_{ijt})$. d_j is a vector of static attributes for potential choice j and γ_i is a vector of parameters related to d_j; y_{jt} is a vector of dynamic attributes for product j at time t and y_{jt} can be energy (typically fuel) cost per mile, purchase price, environment incentives, etc. λ_i is a vector of parameters related to y_{jt}; ε_{ijt} is an individual-specific random term depending of i, the product j and the time period t and it is the component of J-dimensional random vectors ζ_{it}, which are independent and identically-distributed amongst individuals and periods, and zero means. We also require that ζ_{it} follows the generalized extreme value (GEV) distribution, characterized by the cumulative joint distribution function $F_{\zeta}(a_1, ..., a_J)$ of the form $e^{-G(e^{-a_1,...,a_J})}$.

The consumer deciding to buy or postpone is the optimal stopping problem at time t:

$$D(u_{i1t}, ..., u_{iJt}, c_{it}, t) = \max_{\tau} \left\{ \sum_{k=t}^{\tau-1} \beta^{k-t} c_{it} + \beta^{\tau-t} E[\max_{j \in J} u_{ij\tau}] \right\} \tag{3.34}$$

Where β is a discount factor in [0,1). Let $v_{it} = \max u_{ijt}$ and assuming that ν_{it} is Gumbel distributed, the consumer's decision can be transformed from equation (3.34) into:

$$D(v_{it}, c_{it}) = \max\{v_{it}, c_{it} + \beta E[D(v_{i,t+1})]\} \tag{3.35}$$

u_{ijt} can be rewrite with error acting in an additive way: $u_{ijt} = V_{ijt} + \varepsilon_{ijt}$, where V_{ijt} is the mean utility. Relying on McFadden seminal paper (McFadden, 1978), ζ_{it} follows a multivariate extreme value distribution, the choice probabilities can be simplified to usual multinomial logit probabilities.

y_{jt} represents the evolution of the industry and is the dynamic variable, the equation (3.35) can become:

$$D(v_{it}, c_{it}) = \max\left\{v_{it}, c_{it} + \beta E[D\left(v_{i,t+1}(y_{t+1}, c_{i,t+1})\right) | y_t]\right\} \tag{3.38}$$

This is standard optimal stopping problem. The stopping set is given when:

$$T(y_{jt}) = \{v_{it} \mid v_{it} \geq c_{it} + \beta E[D(\cdot) \mid y_t]\} \tag{3.37}$$

$c_{it} + \beta E[D\left(v_{i,t+1}(y_{t+1}, c_{i,t+1})\right) | y_t]$ is called reservation utility $W(y_t)$ when people choose to postpone. Here $y_t = (y_{1t}, ... y_{Jt})$. The reservation utility can be integrated over the industry evolution so that

$$W(y_t) = c_{it} + \beta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max\left(v_{it}, W(z)\right) dF(v \mid z) d\Phi(z \mid y_t) \tag{3.38}$$

The function $W(y_t)$ is a fixed point in a functional space, and its value can be computed using the approach proposed in Rust (1988). Denoting $W(y_t) = \Lambda(W(y_t))$. We need to prove that $\Lambda(\cdot)$ is contractant and to show that $W(y_t)$ is the unique solution. Then use optimization method to calculate $W(y_t)$.

The parameters estimation can therefore be formulated as a traditional maximum likelihood problem:

$$\max_h LL(h) = \sum_{i=1}^{M} \sum_{t=1}^{T} \ln P_{it}[\text{decision} \mid s_{it=0}]. \tag{3.39}$$

The fixed point problem in 3.39 can possibly be solved by applying Rust's nested fixed point algorithm. The fixed point calculation of $W(y_t)$ represents the inner loop. With $W(y_t)$ calculated, the probability function and likelihood function can be calculated and all parameters estimated by maximum likelihood method.

The model can be calibrated on retrospective data about car ownership; however, these are in general aggregate data and the framework above should be reduced to the one adopted by Melnikov (2000) or Carranza (2010). Disaggregate panel data available for car ownership data are usually collected on a yearly basis; we anticipate that a panel on 5-10 years is necessary to estimate a DDCM as in eq. 3.39. The main limitation in using historical data is that the effects of future technology improvements on consumers' behavior cannot be estimated; this is especially true when the characteristics of future cars are expected to change dramatically over time. We are working on the possibility to collect stated preference data with scenarios changing every six months over a total period of six years; the change in the car attributes are supposed to follow a diffusion process as defined in Melnikov (2000).

The framework as proposed in this Section is currently being implemented in an in-house software package programmed in C language.

5 Summary

In this paper we have reviewed Dynamic Discrete Choice Models; these models combine principles derived from random utility theory and dynamic programming. DDCMs were formulized by John Rust in 1987 as an optimal stopping point and applied to a problem with a single agent decision, one purchase and homogenous products. The original framework has been recently improved by considering industry evolution over time, product upgrading, heterogeneity in consumer preferences and repeated purchases over time.

These models have been mainly formulated and applied in economics but are fairly unknown in transportation where discrete choice models are still developed in a static context. However, many decisions in travel behavior are by nature dynamic and researchers are recently attempting to incorporate temporal effects in their models. In particular, we have proposed the application of DDCM to the car ownership problem where in a near future households will be confronted with advanced technology and fast changing fuel prices. This is not the only possible application in transportation, discrete choices to be modeled in revenue management schemes or activity based frameworks are heavily affected by the temporal component.

Possible impediments to the acceptance of DDCM in transportation are the data requirements (panel data) and the relative computational complexity in their application. A number of multi period surveys and continuous travel diaries are now available to researchers and the successful analysis conducted in recent years confirm that panel data are reliable and free from bias due to fatigue effects (Axhausen et al., 2002). The implementation of dynamic programming concepts in current transportation software will encourage the use of DDCM, while their success will certainly depend on the accuracy of short-medium term forecasts.

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